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Passive-based Analysis and Control of Takagi-Sugeno Fuzzy Systems

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Summary

In this paper, sufficient LMI-based conditions for general Quadratic Supply Rate (QSR) functions, known as QSR-dissipativity of nonlinear systems in Takagi-Sugeno form are proposed. To determine the stability of negative feedback loops of two systems, it is sufficient to prove the passivity of each individual system. This property can be used to ensure the stability of even globally distributed systems if the individual local systems are passive. This is particularly relevant for the stabilization of massively distributed power systems, for example. The passivity can either be a property of non-regulated systems or non-passive systems becomes passive through feedback controller. Both cases are analyzed for the class of Takagi-Sugeno fuzzy systems.

1 Introduction

The starting point of this investigation based on the system-theoretical observation that a negative feedback loop consisting of two passive systems is passive, which is a sufficient condition for the stability of negative feedback loops. This means that networked systems comprise passive elements whose negative feedback loops are successively combined to form a passive system. In contrast, negative feedback loops can lead to instability if the individual systems are only stable. This behavior has been observed since the beginning of mathematical investigation of feedback systems in the frequency and time domain. A central concept for passive systems is the consideration of storage functions. Methods that finds a Lyapunov function to prove stability, can also be used to find a storage function. The relationship between dissipativity as a generic concept and Lyapunov stability can be established by employing the storage function $S(x)$ as a Lyapunov function $V(x)$, where $V(x) \geq 0$ for all $x \in X$ is a positive semidefinite function $\mathbb{R}^n \to \mathbb{R}$

$$V(x(t)) - V(x(0)) \leq \int_0^t f_s(u(\tau), y(\tau))\, d\tau \tag{1}$$

with the mathematical formulation of the supply rate $f_s(u(\tau), y(\tau))$, where $u \in \mathbb{R}^m$ denotes the system input and $y \in \mathbb{R}^p$ the system output. The mathematical formulation (1) is called the dissipation inequality. If the supply rate can be described with the bilinear form $f_s(u,y) = u^T y$, the dissipation inequality is related to the definition of passive systems. According to (1), passivity is, therefore, the property that the increase in storage S described mathematically by a storage function is not larger than the bilinear supply rate. Note $f_s(u,y) = u^T y$ requires that the number of system inputs and outputs must be the same. This indicates the appropriate choice of inputs and outputs is already a part of the system analysis and controller synthesis. However, the dissipation inequality (1) is a stronger criterion than the Lyapunov criterion (related to Lyapunov's second method for stability) [5], [7]) which has previously been used in the Takagi-Sugeno (T-S) framework for numerical determination with LMI constraints for synthesis and analysis [2], [3], and [8].

This paper is organized as follows: Section 2 introduces the definition and formulation of the problem, which are taken from the textbooks [9] and [10]. LMI criteria to verify the dissipation properties of LTI and T-S systems are proposed in Section 3. Section 4 deals with the controller design approach to transform non-passive systems into passive systems through appropriate feedback control. An illustrating example is proposed and discussed in Section 5.

2 QSR-Dissipativity and Passivity of Systems

A special form of the supply function in (1) is the general quadratic supply rate (QSR) also known as QSR-dissipativity defined as

$$f_s(u,y) = \begin{pmatrix} y(t) \\ Tu(t) \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} . \tag{2}$$

The choice of the Q-, S- and R-matrices determines the the system category and, with the specified matrix definitions, leads to

- *passive systems* with $Q = 0_{p,p}$, $S = \frac{1}{2}I_p$, and $R = 0_{p,p}$, where $0_{p,p}$ denotes the null matrix and I_p the identity matrix.
- *strictly passive systems* with $Q = -\varepsilon I_p$, $S = \frac{1}{2}I_p$, and $R = -\delta I_p$
  - a distinction is made between *strictly input passive* $\delta > 0$, *strictly output passive* $\varepsilon > 0$, and *very strictly passive* with $\delta > 0$ and $\varepsilon > 0$.
- $\mathcal{L}_2$-*gain systems* with $Q = -I_p$, $S = 0_{p,m}$, and $R = \gamma I_m$

$\mathcal{L}_2$-gain systems meet the norm criterion for the input/output signal given as

$$\underbrace{\int_0^\infty y^T y \, d\tau}_{\|y\|_2^2} \le \gamma \underbrace{\int_0^\infty u^T u \, d\tau}_{\|u\|_2^2} + V(x_0) , \quad x_0 := x(t_0) . \tag{3}$$

In the following section, LMI criteria for LTI and T-S systems will be proposed for analysis and passivity-based controller synthesis.

3 Criteria for General Dissipativity Analysis of LTI and T-S Systems

Derivation of the integral form (1) and the substitution of $f_s(u,y)$ by the quadratic supply rate (2) results in

$$\dot{V}(x(t)) \leq \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}, \tag{4}$$

where $Q = Q^T$ and $R = R^T$. The analysis below is done with the quadratic Lyapunov function candidate

$$V(x(t)) = x^T P x, \quad P \succ 0, \quad P = P^T, \tag{5}$$

$$\dot{V}(x(t)) = \dot{x}^T P x + x^T P \dot{x}. \tag{6}$$

3.1 Dissipativity Analysis of LTI Systems

As a first step, criteria for the dissipativity analysis of LTI state-space systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t) \tag{7}$$

are derived. The substitution of $\dot{x}$ in (6) by the right hand side (rhs) of the state differential equation results in

$$\dot{V}(x) = \big(Ax + Bu\big)^T Px + x^T P\big(Ax + Bu\big). \tag{8}$$

For reasons of simplification, the notation of the time dependency of the variables is omitted. After a brief rearrangement of (8), we obtain

$$\dot{V}(x) = \begin{pmatrix} x \\ u \end{pmatrix}^T \underbrace{\begin{pmatrix} PA + A^T P & PB \\ B^T P & 0 \end{pmatrix}}_{\prec 0} \begin{pmatrix} x \\ u \end{pmatrix} < 0. \tag{9}$$

With (9) as the left hand side of (4), the dissipation inequality in matrix form is obtained

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} PA + A^T P & PB \\ B^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{pmatrix} y \\ u \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} . \tag{10}$$

Using the output equation of (7), the rhs becomes

$$\begin{aligned} \dots \leq &\begin{pmatrix} Cx + Du \\ u \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} Cx + Du \\ u \end{pmatrix} \\ = &\, x^T C^T Q C x + x^T C^T Q D u + x^T C^T S u + u^T S^T C x \\ &+ u^T D^T Q C x + u^T S^T D u + u^T D^T S u + u^T R u + u^T D^T Q D u \\ = &\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} C^T Q C & C^T Q D + C^T S \\ D^T Q C + S^T C & D^T Q D + D^T S + S^T D + R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} . \end{aligned}$$

In combination with the left hand side (lhs) of (10), one obtains

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \underbrace{\begin{pmatrix} PA + A^T P - C^T Q C & PB - C^T Q D - C^T S \\ B^T P - D^T Q C - S^T C & -D^T Q D - D^T S - S^T D - R \end{pmatrix}}_{\preccurlyeq 0} \begin{pmatrix} x \\ u \end{pmatrix} \leq 0 \tag{11}$$

This leads with (5) to the LMI criterion for the general QSR dissipativity of LTI systems in state space form

$$P \succ 0 , \quad \begin{pmatrix} PA + A^T P - C^T Q C & PB - C^T Q D - C^T S \\ B^T P - D^T Q C - S^T C & -D^T Q D - D^T S - S^T D - R \end{pmatrix} \preccurlyeq 0 . \tag{12}$$

Based on the definitions of Q, S, and R in Section 2, the passive, strictly passive and $\mathcal{L}_2$-gain property can be analyzed by solving for a feasible P.

3.2 Dissipativity Analysis of T-S Systems

The results for LTI systems are now to be transferred to T-S systems of the form [8]

$$\dot{x}(t) = \sum_{i=1}^{N_r} h_i(z(t))\big(A_i x(t) + B_i u(t)\big)\,, \quad y(t) = \sum_{i=1}^{N_r} h_i(z(t))\big(C_i x(t) + D_i u(t)\big)\,, \tag{13}$$

where $h_i(z) : \mathbb{R}^l \to \mathbb{R}$ fulfill the convex sum condition

$$\sum_{i=1}^{N_r} h_i(z) = 1, \quad h_i(z) \geq 0\,. \tag{14}$$

The derivation of the quadratic Lyapunov function (6) related to T-S system (13) is given as

$$\dot{V}(x) = \begin{pmatrix} x \\ u \end{pmatrix}^T \sum_{i=1}^{N_r} h_i(z) \begin{pmatrix} PA_i + A_i^T P & PB_i \\ B_i^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}\,. \tag{15}$$

Thus the dissipation inequality (4) for T-S systems is obtained as

$$\sum_{i=1}^{N_r} h_i(z) \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} PA_i + A_i^T P & PB_i \\ B_i^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{pmatrix} y \\ u \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}\,. \tag{16}$$

Substitution of y in rhs of (16) by the output equation (13) yields

$$\begin{aligned}
\dots \leq & \begin{pmatrix} \sum_{i=1}^{N_r} h_i(z)\big(C_i x + D_i u\big) \\ u \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{N_r} h_j(z)\big(C_j x + D_j u\big) \\ u \end{pmatrix} \\
&= x^T \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} h_i(z) h_j(z) C_i^T Q C_j x + x^T \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} h_i(z) h_j(z) C_i^T Q D_j u \\
&\quad + x^T \sum_{i=1}^{N_r} h_i(z) C_i^T S u + u^T S^T \sum_{j=1}^{N_r} h_j(z) C_j x + u^T \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} h_i(z) h_j(z) D_i^T Q C_j x
\end{aligned}$$

$$+u^T S^T \sum_{i=1}^{N_r} h_j(z) D_j u + u^T \sum_{i=1}^{N_r} D_i^T S u + u^T R u + u^T \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} D_i^T Q D_j u \, .$$

Utilizing the convex sum condition (14) of $h_i(z)$ resp. $h_j(z)$ the rhs of (16) results in

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} h_i(z) h_j(z) \begin{pmatrix} C_i^T Q C_j & C_i^T Q D_j + C_i^T S \\ D_i^T Q C_j + S^T C_j & D_i^T Q D_j + D_i^T S + S^T D_j + R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

By using the compact notation proposed in [11], we obtain with the lhs of (16) the inequality

$$\sum_{i=1}^{N_r} \sum_{j=1}^{N_r} h_i(z) h_j(z) \begin{pmatrix} x \\ u \end{pmatrix}^T \Gamma_{ij} \begin{pmatrix} x \\ u \end{pmatrix} \leq 0 \, , \tag{17}$$

where

$$\Gamma_{ij}(P) = \begin{pmatrix} P A_i + A_i^T P - C_i^T Q C_j & P B_i - C_i^T Q D_j - C_i^T S \\ B_i^T P - D_i^T Q C_j - S^T C_j & -D_i^T Q D_j - D_i^T S - S^T D_j - R \end{pmatrix} . \tag{18}$$

With the upper bound, if

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \Gamma_{ij}(P) \begin{pmatrix} x \\ u \end{pmatrix} \leq 0 \tag{19}$$

for each term $i, j = 1, \ldots, N_r$ holds, this also valid for the total sum (17). Finally, utilizing the symmetry of the multiplication of the h_i-functions, the relaxed LMI condition [8] to verify the QSR-dissipativity of T-S system is obtained:

$$\begin{aligned} & P \succ 0 \, , \\ & \Gamma_{ij}(P) + \Gamma_{ji}(P) \preccurlyeq 0 \, , \\ & \Gamma_{ii}(P) \preccurlyeq 0 \quad \text{for all} \quad i = 1, 2, \ldots, N_r \, , \quad j = i+1, i+2, \ldots, N_r \\ & \qquad\qquad s.t. \quad h_i(z) h_j(z) \neq 0, \ \exists z \end{aligned} \tag{20}$$

with $\Gamma_{ij}(P)$ proposed in (18). Based on the definitions of Q, S, and R in Section 2, the passive, strictly passive and $\mathcal{L}_2$-gain properties of nonlinear systems in T-S form (13) can be analyzed by finding a common P. How to apply (20) in principle by solving a convex optimization problem is shown in Section 5 by a toy example. The applicability of (20) to real systems, for example wind and photovoltaic generator models of power systems in T-S form proposed in [11], will be investigated in future studies.

4 Passivity-Based Control of T-S systems

To analyze the passive-based control of a T-S system, the state-space model without direct pass-through is considered:

$$\dot{x} = \sum_{i=1}^{N_r} h_i(z)\big(A_i + B_i u\big)\,, \qquad y = \sum_{i=1}^{N_r} h_i(z)\, C_i x\,. \tag{21}$$

The proposed control law

$$u = \sum_{j=1}^{N_r} h_j(z)\big(K_j x + F_j v\big)\,, \tag{22}$$

consists of a state feedback matrix K_j and gain matrix F_j to feed-forward the reference signal $v \in \mathbb{R}^p$ in order to achieve a steady-state control error of zero. The structure of the control law corresponds to the parallel distributed compensator (PDC) of Takagi-Sugeno fuzzy systems [8], whereby the weighting functions are identical to those of the model equation (21). Obtaining the closed-loop system the input in (21) is substituded by the control law (22). Utilizing the convex sum condition (14) yields to

$$\dot{x} = \sum_{i=1}^{N_r}\sum_{j=1}^{N_r} h_i(z)h_j(z)\big((A_i - B_i K_j)x + B_i F_j v\big)\,. \tag{23}$$

Inserting of (23) into the derivative of the Lyapunov function (6) results directly in

$$\begin{aligned}\dot{V}(x) = & \sum_{i=1}^{N_r}\sum_{j=1}^{N_r} h_i(z)h_j(z)\big((A_i - B_iK_j)x + B_iF_jv\big)^T Px \\ & + \sum_{i=1}^{N_r}\sum_{j=1}^{N_r} h_i(z)h_j(z)\, x^T P\big((A_i - B_iK_j)x + B_iF_jv\big)\ .\end{aligned} \tag{24}$$

After few steps, we obtain

$$\dot{V}(x) = \begin{pmatrix} x \\ v \end{pmatrix}^T \sum_{i=0}^{N_r}\sum_{j=1}^{N_r} h_i(z)h_j(z) \begin{pmatrix} PA_i + A_i^T P - PB_iK_j - K_j^T B_i^T P & PB_iF_j \\ F_j^T B_i^T P & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}\ . \tag{25}$$

As in the analysis of Section 3.2, the design criterion is derived by examining the QSR supply rate in (4):

$$f_s(v,y) = \begin{pmatrix} y \\ v \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$$

Substitution of y by the output equation of (21) yields after few calculation steps

$$f_s(v,y) = \begin{pmatrix} x \\ v \end{pmatrix}^T \sum_{i=1}^{N_r} h_i(z) \begin{pmatrix} C_i^T QC_i & C_i^T S \\ F_j^T B_i^T P - S^T C_i & -R \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}\ . \tag{26}$$

With (25) and (26), the result for the inequality (4) becomes

$$\begin{aligned}&\begin{pmatrix} x \\ v \end{pmatrix}^T \sum_{i=1}^{N_r}\sum_{i=1}^{N_r} h_i(z)h_j(z) \\ &\quad \cdot \underbrace{\begin{pmatrix} PA_i + A_i^T P - PB_iK_j - K_j^T B_i^T P - C_i^T QC_i & PB_iF_j - C_i^T S \\ F_j^T B_i^T P - S^T C_i & -R \end{pmatrix}}_{\Gamma_{ij}(P,K_j,F_j)\preccurlyeq 0} \begin{pmatrix} x \\ v \end{pmatrix}^T \leq 0\end{aligned} \tag{27}$$

Note that the matrix $\Gamma_{ij}(P,K_j,F_j)$ is not linear in P,K_j and F_j. With the restriction that only systems with directly measurable states $y = I_p x$ are to be analyzed regard to passive, strictly passive, and $\mathcal{L}_2$-gain systems we obtain $\Gamma_{ij}(X,X^2,K_j,F_j)$. The matrix is therefore linear in K_j and F_j, but bilinear in X. Steps required for this are explained below. Application of the congruence equivalent relation for (27) gives

$$\Gamma_{ij}(P,K_j,F_j) \preccurlyeq 0 \quad \Leftrightarrow \quad W\Gamma_{ij}(P,K_j,F_j)W^T \preccurlyeq 0 \tag{28}$$

with

$$W = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \quad X := P^{-1},, \quad X = X^T \tag{29}$$

yields

$$W\,\Gamma_{ij}(P,K_j,F_j)\,W^T =$$
$$\begin{pmatrix} A_iX + XA_i^T - B_iK_jX - XK_j^TB_i^T - XC_i^TQC_iX & B_iF_jX - XC_i^TSX \\ XF_j^TB_i^T - XS^TC_iX & -XRX \end{pmatrix}.$$

Utilizing the new variables $M_j := K_jX$, $N_j := F_jX$, where $M_j^T = X^TK_j^T = XK_j$ and $N_j^T = XF_j^T$, we obtain for $C_i = I_p$ the relaxed bilinear matrix inequalities

$$\begin{gathered} X \succ 0\,, \quad \Gamma_{ij}^{\alpha}(X,M_j,N_j) + \Gamma_{ji}^{\alpha}(X,M_j,N_j) \preccurlyeq 0\,, \\ \Gamma_{ii}^{\alpha}(X,M_j,N_j) \preccurlyeq 0 \quad \text{for all} \quad i = 1,2,\ldots,N_r\,, \quad j = i+1, i+2, \ldots, N_r \\ s.t. \quad h_i(z)h_j(z) \neq 0,\ \exists z \end{gathered} \tag{30}$$

with $\alpha = \{a,b,c\}$. The passivity-based controller synthesis for the three types of passivity introduced in Section 2 is formalized by (30) for

- passive systems: $Q = 0_{p,p}$, $S = \frac{1}{2}I_p$, $R = 0_{p,p}$ with

$$\Gamma_{ij}^{a}(X,M_j,N_j) = \begin{pmatrix} A_iX + XA_i^T - B_iM_j - M_j^TB_i^T & B_iN_j - \frac{1}{2}X^2 \\ N_jB_i^T - \frac{1}{2}X^2 & 0 \end{pmatrix} \tag{31}$$

- strictly passive systems: $Q=-\varepsilon I_p, S=\frac{1}{2}I_p, R=-\delta I_p$ with

$$\Gamma_{ij}^{b}(X,M_j,N_j)=\begin{pmatrix} A_iX+XA_i^T-B_iM_j-M_j^TB_i^T+\varepsilon X^2 & B_iN_j-\frac{1}{2}X^2 \\ N_j^TB_i^T-\frac{1}{2}X^2 & \delta X^2 \end{pmatrix} \tag{32}$$

- $\mathcal{L}_2$-gain systems $Q=-I_p, S=0_{p,m}, R=\gamma I_m$ with

$$\Gamma_{ij}^{c}(X,M_j,N_j)=\begin{pmatrix} A_iX+XA_i^T-B_iM_j-M_j^TB_i^T+X^2 & B_iN_j \\ N_j^TB_i^T & -\gamma X^2 \end{pmatrix} \tag{33}$$

Note the state feedback gain K_j and feed forward gain F_j for $j=1,2,..,N_r$ of the PDC control law (22) are calculated by $K_j=M_jX^{-1}$ and $F_j=N_jX^{-1}$.

5 Illustrating Example

Finally, after formal consideration of the QSR-dissipativity applied to T-S systems, an illustrative mechanical toy example is examined. This was selected in such a way that an analytical solution is also available. Given is a mechanical oscillator with one degree of freedom shown in Figure 1. The displacement of the rigid body with $m=2$ (kg) is restricted to the direction x (m), whereby the stiffness in the mechanical system is described via the spring $c=2$ (N/m) and the damping is described with the constituent formula

$$d(\dot{x})=d_0+d_1\dot{x}^2 \tag{34}$$

with the parameters $d_0=0.1$ (N/m/s) and $d_1=0.05$ (N/m^2/s^2). The state space model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{c}{m}x_1-\frac{d_0}{m}x_2-\frac{d_1}{m}x_2^3+\frac{1}{m}u\,, \qquad x_0=x(0)\,, \\ y &= Cx \end{aligned} \tag{35}$$

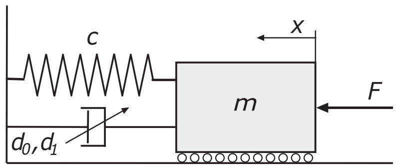

Figure 1: Mechanical Oscillator with nonlinear damper as an illustrating example

with $x = (x_1\,, x_2)^T := (x\,, \dot{x})^T$ and $u := F$ provides a complete description of the system dynamics. Let us first examine the passivity property of (35) analytically by using the dissipation inequality (1) directly i.e. without the proposed LMI formulation. The following Lyapunov function candidate is used for the example:

$$V(x) = \frac{1}{2} c x_1^2 + \frac{1}{2} m x_2^2 > 0 \quad \text{for} \quad x_1, x_2 \neq 0$$
$$\dot{V}(x) = c x_1 \dot{x_1} + m x_2 \dot{x_2}$$

Substituting the derivatives with the right hand sides of (35) results in

$$\dot{V}(x) = c x_1 x_2 + m x_2 \big(-\frac{c}{m} x_1 - \frac{d_0}{m} x_2 - \frac{d_1}{m} x_2^3 + \frac{1}{m} u \big) = -d_0 x_2^2 - d_1 x_2^4 + x_2 u\,.$$

Note, if $u = 0$ then $\dot{V}(x) < 0$ for all $x_2 \neq 0$ holds, this means that $V(x)$ is a Lyapunov function and the eigen-motion of (35) is asymptotically stable [5].

With an input $u \neq 0$ the derivative of dissipation inequality (1) results in

$$\dot{V}(x) \leq f_s(u, y) = u^T y\,. \tag{36}$$

Choosing the supply rate of $f_s(u, y) = u^T y$, the fulfillment of the inequality (36) directly results in the evidence of system passivity. Let us consider two orthogonal cases to investigate the passivity of the mechanical oscillator. In the first case only the position x and in the second case the velocity $\dot{x}$ can be measured, i.e. the output matrix C in (35) is

- Case 1: $C = (1\,, 0) \quad \Rightarrow \quad y = x_1 = x\,, \quad f_s(u, y) = u x_1$

- Case 2: $C = (0\,,\,1) \quad \Rightarrow \quad y = x_2 = \dot{x} \quad f_s(u,y) = u x_2$

Thus, for the first case the dissipation inequality (36) for passive systems is

$$-d_0 x_2^2 - d_1 x_2^4 + x_2 u \;\leq\; u x_1 \;\Rightarrow\; \text{ infeasible problem for } x_1, x_2 \in \mathbb{R}$$

and for the second case

$$\begin{aligned} -d_0 x_2^2 - d_1 x_2^4 + x_2 u \;&\leq\; u x_2 \\ -d_0 x_2^2 - d_1 x_2^4 \;&\leq\; 0 \Rightarrow \text{ feasible problem for } x_1, x_2 \in \mathbb{R} \end{aligned}$$

The results are independent from the values of d_0 and d_1. Furthermore, the choice of output has a significant influence on the system characteristics. This means that the system properties of passivity not only depend on the input to state mapping but also on the effect of internal systems states on the output. After the analytical investigation, the LMI-based criteria is now implemented by numerical convex optimization. First the nonlinear system is presented as T-S system. For this purpose, the nonlinear sector approaches is used, where T-S models exactly represent nonlinear systems in a bounded state space. In the case study, the nonlinearity is determined by the interval of $z \in [z_{min}, z_{max}]$ with $z := \dot{x}^2$, $z_{min} = 0$ and $z_{max} = \dot{x}_{max}^2$. The damping function $d(z) = d_0 + d_1 z$ proposed in (34) with the new coordinate z can therefore be substituted by the following sector function

$$d(z) = \frac{d_{max} - d(z)}{d_{max} - d_{min}}\, d_{min} + \frac{d(z) - d_{min}}{d_{max} - d_{min}}\, d_{max} = h_1(z)\, d_{min} + h_2(z)\, d_{max} \qquad (37)$$

with $d(z) \in [d_{min}, d_{max}]$, where $d_{min} = d_0$ and $d_{max} = d_0 + d_1\, z_{max}$. The membership functions of the T-S models result from the z-dependent weightings of the sector bounds in (37)

$$h_1(z) = \frac{d_{max} - d(z)}{d_{max} - d_{min}}\,, \qquad h_2(z) = \frac{d(z) - d_{min}}{d_{max} - d_{min}}\,. \qquad (38)$$

The state space model of the mechanical oscillator (35) then results as follows

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d(z)}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix} u \\ &= \begin{pmatrix} 0 & 1 \\ -\frac{c}{m} & -h_1(z)\,\frac{d_{min}}{m} - h_2(z)\,\frac{d_{max}}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix} u \,.\end{aligned}$$

Utilizing the convex sum property (14), here with $h_1(z) + h_1(z) = 1 \;\; \forall z$, we get the final T-S representation of (35) to

$$\dot{x} = h_1(z) \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{c}{m} & \frac{d_{min}}{m} \end{pmatrix}}_{A_1} x + h_2(z) \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{c}{m} & \frac{d_{max}}{m} \end{pmatrix}}_{A_2} + \underbrace{\begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix}}_{B} u \,. \tag{39}$$

and in the standard sum form with the output equation

$$\dot{x} = \sum_{i=1}^{2} h_i(z) A_i + Bu, \qquad y = Cx \,. \tag{40}$$

Note that B, C, D are common and $D = 0$. That significantly reduce the number of LMIs to verify the QSR-dissipativity by (20) of T-S systems. With the specification $Q = 0_{p,p}$, $S = \frac{1}{2} I_p$, $R = 0_{p,p}$, which is used to examine only the passivity as in the analytical analysis, we obtain the reduced condition to be verified

$$\begin{aligned} &P \succ 0 \,, \\ &\Gamma_i(P) \preccurlyeq 0 \quad \text{for all} \quad i = 1, 2, \ldots, N_r \,, \end{aligned} \tag{41}$$

where

$$\Gamma_i(P) = \begin{pmatrix} PA_i + A_i^T P & PB - \frac{1}{2} C^T \\ B^T P - \frac{1}{2} C & 0 \end{pmatrix} . \tag{42}$$

Using the numerical values of the model parameters given above, the following results for (41), (42) were obtained with the SDPT3-4 solver within the YALMIP toolbox (version 20210331), Matlab R2020a:

- Case 1: $C = (1\,,0)$ $\Rightarrow$ `Infeasible problem (SDPT3-4)`
- Case 2: $C = (0\,,1)$ $\Rightarrow$ `Successfully solved (SDPT3-4)`

The result refers to the sector limit $z_{max} = \dot{x}^2_{max}$ with upper bound $\dot{x}_{max} = 2$. But could also be confirmed for larger values such as $\dot{x} = \{2,4,\ldots,18,20\}$ i.e. the distance between the linear models in the parameter space is up two orders larger.

6 Conclusion

The QSR-dissipativity for the class of T-S fuzzy systems was investigated. An LMI-based verification and a BMI-based design approach for passive-based control were presented. Studies on synthesis with BMIs have yet to be carried out. A two-step procedure could convert the latter BMI into an LMI formulation. First, a Lyapunov function is calculated with the left hand side of the standard dissipation inequality, and in the second step, X^2 is calculated using an auxiliary variable. What could be shown in closing is that the proof of passivity in T-S fuzzy systems is numerically possible by means of convex optimization algorithms as implemented in [6]. Furthermore, because only simple test models have been used to date, more realistic physical process models [11], such as those used to verify the stability of distributed power systems, are also being investigated using the proposed approach.

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